

Fermi Questions

Fermi Questions is named after Enrico Fermi, a Nobel Laureate in Physics, who was famed for being able to do order-of-magnitude calculations in his head. For example, after watching the first atomic bomb explosion, he immediately calculated that the strength of the explosion was equivalent to the explosion of 20 kilotons of TNT. It took another three weeks for a panel of the Manhattan Project's best scientific brains to do an 'exact' calculation; the answer that they came up with, yes, you guessed it - 20 kilotons. Such calculations are sometimes called 'ballpark estimates' or 'back-of-the-envelope' calculations. While these calculations were important, years ago, because one had to keep track of decimal places when using a slide rule, these calculations are still very important because an approximate answer will often dictate the amount of resources required to attack a problem. For example, when you consult a wedding consultant to plan the affair, they often ask the question, "How many people will attend the dinner?". Your approximate answer will allow them to estimate the amount of food required, the number of tables and their layout, the size of the hall to be rented, etc., etc. Fundamental to the solution of these problems is a skill called Critical Thinking - essentially a method of attacking such problems in an orderly, logical way. This skill can be learned and it is the underlying basis for the event.

Why this event? Numbers (when you think about it) are a measure of our surroundings and life.

There are several types of questions that can be answered by this procedure:

- * math (straight) – where the answer can be calculated using a calculator or computer but, since such aids are not allowed in the competition, it forces the student to consider other routes to provide a reasonable answer

- * how answers from one problem relate to other problems – as with many facets of life, an answer to one problem leads to many other choices and problems.

- * having solutions to problems relate to 'real life', for example, a problem might ask for an estimate of the amount of gasoline used by passenger cars in the U.S., how an increase in gas mileage would relate to a decrease in green-house gas production, and how the amount of water produced by same relates to other items such as rainfall or filling of swimming pools.

In short, if something has numbers associated with it, that subject is fair game for a Fermi Question.

Underlying Considerations

- * Behind each problem set that I create is the tacit assumption that the contestants have a reasonable knowledge of mathematics, specifically, the use and operations of exponential notation. The lack of math skills is not too apparent when the answers to the problems are in the range 0.001 to 1000 (Fermi Question notation -3 to $+3$). But when I ask the students to calculate the number of iron atoms on the head of a pin, the inability to handle exponents readily shows (there are approximately 3×10^{13} iron atoms – Fermi Question answer $+13$; see Example xiv. below). I can't count the number of times that I've seen students cover the scrap paper (that I distribute for them to use in their deliberations) with zeroes. For that reason, it is imperative to stress the use of exponential notation (which also serves as the basis for the metric system). Not only does the use of exponential notation make calculations faster, but it also helps avoid problems with writing, transcribing, and counting the correct number of zeroes. So much so, that in some branches of science there are specially named units that have very large (or very small) numbers associated with them, such as, one Angstrom = 10^{-8} cm, one Light Year = 5.9×10^{12} miles, Avogadro's Number = 6.023×10^{23} .

* As I noted previously, an important component of the event is logical, critical thinking. Reading and understanding the problem is one important component; the other important component is to develop a plan to provide the answer in the requested units.

* And finally, time is a critical parameter. The ability to think and calculate rapidly can be learned – the keywords are, in the immortal words of a Hall-of-Fame football coach, "practice, practice, and more practice". I have watched students (when I was a coach) significantly lower the times required to solve these problems. In fact, some of them have returned from college and told me that the same skills (required to solve Fermi Questions) permitted them to handle tests and problem sets much faster than their contemporaries.

Typically, the first time that a team tries to solve a problem, they try to be too exact. For example, if the Fermi Question is "how many toothpicks are equivalent to the perimeter of Colorado?", they discuss the length of a toothpick ("is it 2.0, 2.25, 2.45 inches?"); then they try to estimate the perimeter of the state; and finally, they calculate a value. Any time there is a discussion, time is lost. Since the answer to any question is the correct order of magnitude, an error of a factor of two or three will probably yield the correct exponent (the Fermi Question answer). Hence, they should pick a value and work up their answer. The time that they save will be needed to solve other problems.

Examples

- how many air molecules are in this room (where I was presenting this lecture)?
- how many pounds of CO₂ and H₂O does the U.S. population expel in a year?
- how many tons of food are consumed in Chicago during the course of a day?
- how many people are involved in delivering and preparing that food?
- how many gallons of paint do you need to paint the walls of your school?
- how many baseballs are used during the course of a Major League season?
- how many pizzas were eaten last year in the U.S.?

Scoring

The scoring for the event is like horseshoes:

5 points for the correct exponent

3 points for the correct exponent ± 1

1 point for the correct exponent ± 2

The answer to a Fermi Question is the correct exponent of 10 (if an answer is $5 \cdot 10^n$, round the answer up to the next power of 10; I try to manage the problems so that answers are not $5 \cdot 10^n$). Generally, if a team averages 3 points per problem and there are 30 problems, the 90 points that they will have achieved will garner them first or second place. Calculators, computers, or any other device, including crib sheets, lists of constants, formulae, etc., are not permitted. All the contestants need are pencils (with erasers) and a good night's sleep – The Olympiad supplies scratch paper (to simulate the 'back-of-envelopes'). Positive exponential values are the default; negative exponents MUST have the - (minus) sign as part of the answer.

Some considerations involved when learning to solve these problems:

1. **Exponents are short-hand notation** (knowledge of which makes it easier and faster to solve the problems). The notation used below is: Ex. = example; Ans. = Answer; FA = Fermi Answer.

Ex. What is the population of New York City? Ans. $7,000,000 = 7 \cdot 10^6 \sim 10^7$ FA 7

Ex. What is the distance, in miles, from the Earth to the Sun? Ans. $100,000,000 = 10^8$ FA 8

2. Properties of exponents.

$500 = 5 \cdot 10^2$; 5 is the coefficient, 10 is the base, 2 is the exponent
when multiplying, add exponents of the same base

Ex. $200 \cdot 4000 = 2 \cdot 10^2 \cdot 2^2 \cdot 10^3 = 2^3 \cdot 10^5 = 8 \cdot 10^5 \sim 10^6$ FA 6

when dividing, subtract exponents of the same base

Ex. $200 \div 800 = 2 \cdot 10^2 \div 2^3 \cdot 10^2 = 2^{-1} \cdot 10^0 = 2^{-1} \cdot 1 \sim 10^{-1}$ FA -1

note that $10^0 = 1$

$10^{20} = (10^4)^5 = (10^2)^{10}$; $2^{10} \sim 10^3$

3. **Round off values BEFORE doing a calculation.** This makes it much easier and faster to do the problems. Why? because the FA is the correct order of magnitude which means that there is a large range that yields the correct answer. For example, the FA for the distance, in miles, from the Earth to the Sun is 8 (shown previously in 1.) but the range of values giving the same answer is $5 \cdot 10^7$ to $4.99 \cdot 10^8$!! In this context, I suggest using the values below which are somewhat different from the exact values:

Item	Exact Value	Fermi Value (for ease of calculation)
1 day	24 hours	25 hours
1 mile	5280 feet	5000 feet
1 yard	0.9144 meter	1 meter
1 foot	30.48 cm	30 cm
1 pound	453.6 g	500 g
1 hour	3600 seconds	4000 seconds

4. **Always keep the units as part of working a problem.** In some instances, keeping track of the units will lead to the correct answer. The U.S. uses both metric and English systems of units. Sometimes the units get left off solutions to real problems with tragic, unforeseen results. As an example, most US cooks know what a $\frac{1}{4}$ pound of butter looks like - it is a stick about 1 inch x 1 inch x 5 inches. But ask them what 100 g of butter looks like and they may throw up their hands in defeat. The answer is that the stick is almost the same size since 100 g is close to $\frac{1}{4}$ pound.

5. **What subject matter is covered?** Everything is fair game! If the item in question has numbers associated with it, it might be used. On the past tests, questions have been used from math, chemistry, physics, biology, geology, geography, economics, swimming, basketball, running, census, food, waste generation, ...

Examples. (abbreviation: F?s = Fermi Question solution) These can be done by the students as practice; have them show all work and what assumptions they made in solving the problems.

i. How many seconds are there in a year?

Exact solution: $60 \text{ sec/min} \cdot 60 \text{ min/hr} \cdot 24 \text{ hr/day} \cdot 365 \text{ day/year} = 3.15 \cdot 10^7 \text{ s/y}$ FA 7

F?s: $4000 \text{ s/h} \cdot 25 \text{ h/d} \cdot 400 \text{ d/y} = 4 \cdot 10^3 \cdot 2.5 \cdot 10^1 \cdot 4 \cdot 10^2 = 4 \cdot 2.5 \cdot 4 \cdot 10^6 = 4 \cdot 10^7$ FA 7

Note that both answers are the same. Budding Fermi Question experts should remember that there are $3 \cdot 10^7$ seconds in a year - this will probably save them time in solving another F?.

ii. How many miles are there in a light-year?

Exact solution: $186,000 \text{ m/s} \cdot 3.15 \cdot 10^7 \text{ s/y} = 1.86 \cdot 3.15 \cdot 10^{12} = 5.9 \cdot 10^{12} \text{ m/y}$ FA 13

F?s: $2 \cdot 10^5 \text{ m/s} \cdot 3 \cdot 10^7 \text{ s/y} = 6 \cdot 10^{12} \text{ m/y}$

This quantity is a basic unit used in astronomy. As noted in problem i., knowing that there are $3 \cdot 10^7$ seconds in a year has shortened the work considerably.

iii. How many kilometers are there in a light-year?

F?s: $6 \cdot 10^{12} \text{ mi/y} \cdot 1.6 \text{ km/mi} = 10 \cdot 10^{12} \text{ km/y} = 10^{13}$ FA 13

iv. For the average American woman, how many times will her heart beat during her lifetime?

Assumptions: 1 heartbeat per second, lifetime of 80 years

F?s: $3 \cdot 10^7 \text{ s/y} \cdot 1 \text{ hb/s} \cdot 80 \text{ y/lifetime} = 2.4 \cdot 10^9$ FA 9

- v. How many heartbeats are there in a year for the entire world's population?
 Assumptions: 1 heartbeat per second, $6 \cdot 10^9$ people
 $F?s = 3 \cdot 10^7 \text{ s/y} * 1 \text{ hb/s} * 6 \cdot 10^9 = 18 \cdot 10^{16} = 1.8 \cdot 10^{17}$ FA 17
- vi. How many pounds of rice were consumed in the U.S. in the year 2001?
 Assumptions: 20 pounds of rice eaten per year by a person, $3 \cdot 10^8$ people in the U.S.
 $F?s: 20 \text{ \#/p} * 3 \cdot 10^8 \text{ p} = 6 \cdot 10^9 \text{ \#} = 10^{10}$ FA 10
- This answer was checked using data from the U.S. Dept. of Agriculture; $5.2 \cdot 10^9$ lbs. If the students assume 2-10 or 200-1000 #/p, they would still get 3 points.
- vii. What is the density of butter in g/cc?
 Assumptions: 1 pound of butter is a package 2 inch x 2 inch x 5 inches.
 $F?s: V = 2 \text{ in} * 2.5 \text{ cm/in} * 2 \text{ in} * 2.5 \text{ cm/in} * 5 \text{ in} * 2.5 \text{ cm/in} = 5 * 5 * 12 = 300 \text{ cm}^3$
 Density = $M/V = 500 \text{ g} / 300 \text{ cm}^3 = 1.5 \text{ g/cm}^3 \sim 1 = 10^0$ FA 0
- viii. What fraction of a mile is a cm?
 $F?s: 1 \text{ cm} / (5000 \text{ f/mi} * 30 \text{ cm/f}) = 1 / 3 * 5 * 10^4 = 1 / 1.5 * 10^5 = 10^{-5}$ FA -5
- ix. What is the area, in sq. miles, of the original 48 states of the U.S.?
 Assumption: the U.S. is shaped like a rectangle; 3000 miles wide x 1000 miles
 $F?s: 3 \cdot 10^3 * 1 \cdot 10^3 = 3 \cdot 10^6$ FA 6
- Note: when areas are requested, it is much easier to use a rectangle.
- x. What is the area of the U.S. (prob. ix.) in cm^2 ?
 $F?s: 3 \cdot 10^6 \text{ mi}^2 * 1.6^2 \text{ km}^2/\text{mi}^2 * (10^3)^2 \text{ m}^2/\text{km}^2 * (10^2)^2 \text{ cm}^2/\text{m}^2$
 $= 3 * 2.5 * 10^{(6+6+4)} = 7.5 * 10^{16} \sim 10^{17}$ FA 17
- xi. What is the area of Lake Superior in sq. miles?
 Assumption: the lake is shaped like a rectangle; 300 miles wide x 100 miles
 $F?s: 3 \cdot 10^2 \text{ mi} * 1 \cdot 10^2 \text{ mi} = 3 \cdot 10^4$ FA 4
- xii. Estimate the volume of Lake Superior in cubic kilometers.
 $V = \text{Area} * \text{Depth}$; Assumption: Average depth is 200 m
 $F?s: 3 \cdot 10^4 \text{ mi}^2 * 1.6^2 \text{ km}^2/\text{mi}^2 * 200 \text{ m} * 1 \text{ km}/1000 \text{ m}$
 $= 3 * 2.5 * 2 * 10^{(4+2-3)} = 15 * 10^3 = 1.5 \cdot 10^4$ FA 4
- xiii. How many cubic kilometers of rain fall of the U.S.(48 states) in one year? Assume an average rainfall of 10 inches.
 $F?s: 3 \cdot 10^6 \text{ mi}^2 * 1.6^2 \text{ km}^2/\text{mi}^2 * 10 \text{ in} * 2.5 \text{ cm/in} * 1 \text{ m}/10^2 \text{ cm} * 1 \text{ km}/10^3 \text{ m}$
 $= 3 * 2.5 * 2.5 * 10^{(6+1-2-3)} = 20 * 10^2 = 2 * 10^3$ FA 3
- Note that Lake Superior has 7 times the total volume of rainfall: the Great Lakes have about half of the Earth's fresh water.
- xiv. How many iron atoms are on the head of a pin?
 Assumption: the head is 1 mm in diameter, diameter of an iron atom is 2.5 Angstroms
 Area of the head of a pin: $\frac{1}{2} * \pi * D^2 = 1.5 * (1 \text{ mm} * 1 \text{ cm}/10 \text{ mm})^2 = 1.5 * 10^{-2} \text{ cm}^2$
 Assume that the head of a pin is half a sphere
 Area covered by an iron atom: $\frac{1}{4} * \pi * D^2 = 0.75 * (2.5 \cdot 10^{-8})^2 \text{ cm}^2 = 5 * 10^{-16} \text{ cm}^2$
 $F?s: 1.5 \cdot 10^{-2} \text{ cm}^2 / 5 \cdot 10^{-16} \text{ cm}^2 = 0.3 \cdot 10^{14} = 3 * 10^{13}$ FA 13
- Note: this problem can also be solved using the rectangle approach.
 Assumption: the head is 1 mm on a side, an iron atom is 2.5 Angstroms on a side
 Area of the head of a pin: $\frac{1}{2} * 6 * S^2 = 3 * (1 \text{ mm} * 1 \text{ cm}/10 \text{ mm})^2 = 3 * 10^{-2} \text{ cm}^2$
 Area covered by an iron atom: $S^2 = (2.5 \cdot 10^{-8})^2 \text{ cm}^2 = 6.25 * 10^{-16} \text{ cm}^2$
 $F?s: 3 \cdot 10^{-2} \text{ cm}^2 / 6.25 \cdot 10^{-16} \text{ cm}^2 = 0.4 \cdot 10^{14} = 4 * 10^{13}$ FA 13